Exploring a Stronger Triangle Inequality

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Elementary geometry points out that in any triangle, the sum of two sides is always greater than the third side. This study seeks to determine and show when the stronger triangle inequality

$$a + b > c + h$$

(1)

holds. The study will show, as Bailey and Bannister did, that it does indeed hold for most triangles, when the *distinguished angle*, the angle opposite the longest side, is less than $\frac{\pi}{2}$. Further exploration will take place to determine what "most" means, as well as considering what happens when h is divided by 2.

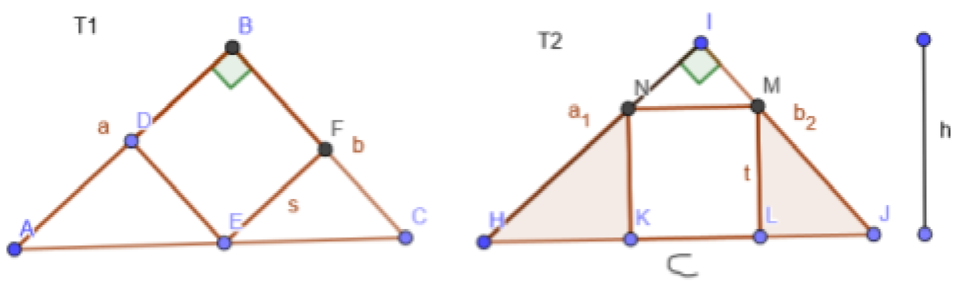

Figure 1

In [1], the *inscribed square problem* is explored first. Squares that have sides of *s* and *t* are inscribed in a right triangle with sides *a, b*, hypotenuse *c*, and altitude *h.* (Note: $\Delta ABC \cong \Delta HIJ$). This begs to question, which square is larger? First, create a new triangle in Figure T2 by combining the shaded areas. Notice that $\Delta HNJ \sim \Delta HIJ$and $\Delta ABC \sim \Delta EFC$by AA. This yields:

$$\frac{s}{b-s} = \frac{a}{b}\text{implies } s = \frac{ab}{a+b}\text{and } \frac{t}{c-t} = \frac{h}{c}\text{implies } t = \frac{hc}{c+h}$$

As $ab = hc$(twice the area), the denominators can be compared. The authors in [1] set:

$$D := (a + b)^2 - (c + h)^2 = a^2 + 2ab + b^2 - c^2 - 2hc - h^2$$

And since $a^2 + b^2 = c^2,\ ab = hc$,it can be concluded that $D =- h^2 < 0$.Which implies that $a + b < c + h$,so $s > t$, showing that inequality (1) is false for any right triangle.

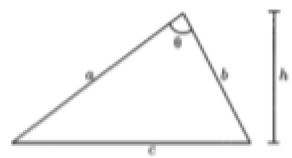

Figure 2

Bailey and Bannister then explored the inequality, with $\theta < \frac{\pi}{2}$,where $\theta$is the angle opposite the longer side *c* and *h* is the height that corresponds to side *c*. A similar argument is explored here. $a^2 + b^2 = c^2$by Pythagorean Theorem, $c^2 = a^2 + b^2 - 2acos\theta$by the law of cosines, and $\frac{1}{2}ch = absin\theta$,expressing the area of the triangle in two different ways.

This yields: $$D := (a + b)^2 - (c + h)^2 = a^2 + 2ab + b^2 - c^2 - 2hc - h^2$$

$$= 2abcos\theta + 2ab - 2absin\theta - \frac{(absin\theta)^2}{c^2}$$

$$= ab(2 + 2cos\theta - 2sin\theta - \frac{absin^2\theta}{a^2+b^2-2abcos\theta})$$

Setting $g(\theta) = 2(1 + cos\theta - sin\theta)$and $R = \frac{a}{b}$

$$D = ab[g(\theta) - \frac{Rsin^2\theta}{R^2+1-2Rcos\theta}]$$

Since $ab\ and\ R^2 + 1 - 2Rcos\theta$are positive, inequality $a + b > c + h$holds if and only if

$$\frac{D(R^2+1-2Rcos\theta)}{ab} = g(\theta)R^2 - [2g(\theta)cos\theta + sin^2\theta]R + g(\theta)$$

$$F(\theta, R) = g(\theta)R^2 - [2g(\theta)cos\theta + sin^2\theta]R + g(\theta)$$

Points of interest can be found by setting $F(\theta, R) = 0$. When $g(\theta) = 0, \theta = \frac{\pi}{2} or\ \pi$. And solving $F(\theta, R)$for R, to get the point $P_3(\frac{\pi}{2}, 0)$ and the line $\theta = \pi$.In the paper, the authors utilized a CAS to factor when $g(\theta) \neq 0$,as $F(\theta, R) = 0$is a quadratic in R. This can be done algebraically and while the factoring may be a little different, the same points can be explored. Let:

$$a = g(\theta),\ b = 2g(\theta)cos\theta + sin^2\theta\ and\ c = g(\theta)$$

Then the discriminant is $D = [2g(\theta)cos\theta + sin^2\theta]^2 - 4g(\theta)^2$

Set $D = 0$and notice that D is of the form $A^2 - B^2 = (A - B)(A + B)$

This yields:

$$0 = [2g(\theta)cos\theta + sin^2\theta - 2g(\theta)][2g(\theta)cos\theta + sin^2\theta + 2g(\theta)]$$

(2)

Focusing on the left side of (2):

$$0 = 4cos\theta + 4cos^2\theta - 4cos\theta sin\theta + sin^2\theta - 4 - 4cos\theta + 4sin\theta$$

The first and sixth terms cancel out.

The second and fifth terms are equivalent to: $4cos^2\theta - 4 =- 4sin^2\theta$

This yields:

$$=- 4cos\theta sin\theta - 3sin^2\theta + 4sin\theta$$

$$= sin\theta(-\ 4cos\theta\ -\ 3sin\theta\ +\ 4)$$

Focusing on the right side of (2):

$$= 4cos\theta\ +\ 4cos^2\theta\ -\ 4cos\theta sin\theta\ +\ sin^2\theta\ +\ 4\ +\ 4cos\theta\ -\ 4sin\theta$$

(3)

Factor out $1\ +\ cos\theta$ in the first and second terms to get $4cos\theta(1\ +\ cos\theta)$

The fourth term can be written as $sin^2\theta\ =\ 1\ -\ cos^2\theta\ =\ (1\ +\ cos\theta)(1\ -\ cos\theta)$

In the third and seventh term, factor out $1\ +\ cos\theta$to get $-\ 4sin\theta(1\ +\ cos\theta)$

The fifth and sixth terms can be factored to get$4(1\ +\ cos\theta)$

Together this yields:

$$= (1\ +\ cos\theta)(4cos\theta\ -\ 4sin\theta\ +\ 4\ +\ 1\ -\ cos\theta)$$

$$= (1\ +\ cos\theta)(3cos\theta\ -\ 4sin\theta\ +\ 5)$$

(4)

(2) is equivalent to (3) and (4) together:

$$0\ =\ (1\ +\ cos\theta)(sin\theta)(-\ 4cos\theta\ -\ 3sin\theta\ +\ 4)(3cos\theta\ -\ 4sin\theta\ +\ 5)$$

Our points of interest take place when $D\ =\ 0$,so setting each parenthesis equal to 0:

$0\ =\ 1\ +\ cos\theta$ therefore $\theta\ =\ \pi$

$0\ =\ sin\theta$ therefore: $\theta\ =\ 0,\ \pi$

$0\ =-\ 4cos\theta\ -\ 3sin\theta\ +\ 4$ solve for θ:

$4(1\ -\ cos\theta)\ =\ 3sin\theta$

$16(1\ -\ 2cos\theta\ +\ cos^2\theta)\ =\ 9sin^2\theta$but $9sin^2\theta\ =\ 9\ -\ cos^2\theta$

$25cos^2\theta\ -\ 32cos\theta\ +\ 7\ =\ 0$

Notice the quadratic in relation to $cos\theta$,$D = 32^2 - (4 * 7 * 25) = 4 * 81$

So $cos\theta = \frac{32\pm18}{50}$yields $cos\theta = 1\ or\ \frac{7}{25}$

Through substitution, $sin\theta = \frac{4}{3}(1 - \frac{7}{25}) = \frac{24}{25}$

therefore $\theta = 0,\ arctan\ \frac{24}{7}$

$0 = 3cos\theta - 4sin\theta + 5$

$4sin\theta = 3cos\theta + 5$

$16sin^2\theta = 9cos^2\theta + 30cos\theta + 25$

$16 - 16cos^2\theta = 9cos^2\theta + 30cos\theta + 25$

$0 = 25cos^2\theta + 30cos\theta + 9$

Notice the quadratic in relation to $cos\theta$,$D = 30^2 - (4 * 25 * 9) = 0$,so

$cos\theta = \frac{-3}{5}$

Through substitution, $4sin\theta = 3(\frac{-3}{5}) + 5andsin\theta = \frac{4}{5}$

therefore $\theta = \pi - arctan\ \frac{4}{3}$

By substituting the values back into $F(\theta, R)$,solve for R, the following points of interest can be found. $P_1(0, 1)$, $P_2(arctan\ \frac{24}{7}, 1)$, $P_4(\pi - arctan\ \frac{4}{3}, - 1)$, $P_5(\pi, - 1)$.Together with $P_3(\frac{\pi}{2}, 0)$ and the line $\theta = \pi$,the function, and graph, can be analyzed more closely.

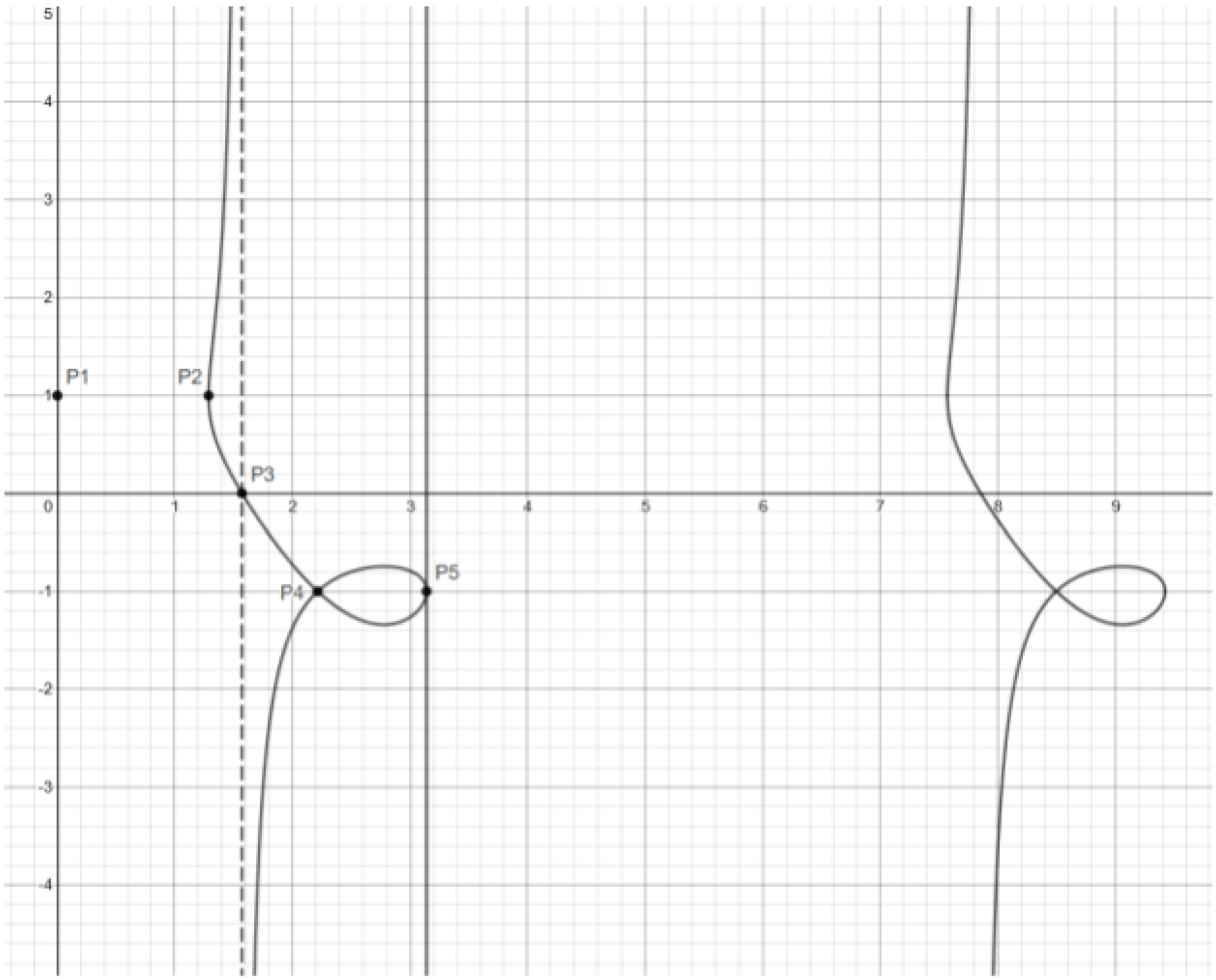

Figure 3

As [1] explains:

"The zeros of this discriminant occur at the four points $P_1(0, 1)$,$P_2(arctan\ \frac{24}{7}, 1)$, $P_4(\pi - arctan\ \frac{4}{3}, - 1)$,and $P_5(\pi, - 1)$.These points, along with $P_3(\frac{\pi}{2}, 0)$ and the asymptote at $\frac{\pi}{2}$are shown. The upper and lower branches of the quadratic meet at P_2, P_3, P_4, and P_5. The point $P_1(0,1)$ is an isolated solution for $F(\theta, R) = 0$. From the coordinates

$P_2(arctan\,\frac{24}{7}, 1)$we conclude that any triangle in which the angle θ between sides *a* and *b* is less than $arctan\,\frac{24}{7} \approx 74^o$will satisfy the strong triangle inequality, and an isosceles triangle with $\theta = arctan\,\frac{24}{7}$provides a counterexample to the inequality with the smallest possible value of θ."

What does "most" mean?

The stronger triangle inequality depends on θand it is of interest to determine how often it holds, given that $\theta < \frac{\pi}{2}$. Faiziev, Powers, and Sahoo provided the following argument, which was based upon the equation proven by Klamkin in [3]: Suppose one is given an ordered pair (α, β), that represents the two other angles in the triangle, then $\theta + \alpha + \beta = \pi$.The validity of the stronger triangle inequality does not change with respect to similarity, so all triangles with $\theta < \frac{\pi}{2}$ are those with ordered pairs $(\alpha, \beta) \in R^2$such that:

$0 < \alpha < \pi, 0 < \beta < \pi$, and $-\alpha + \frac{\pi}{2} < \beta < -\alpha + \pi$.

A sketch of *R* is shown below.

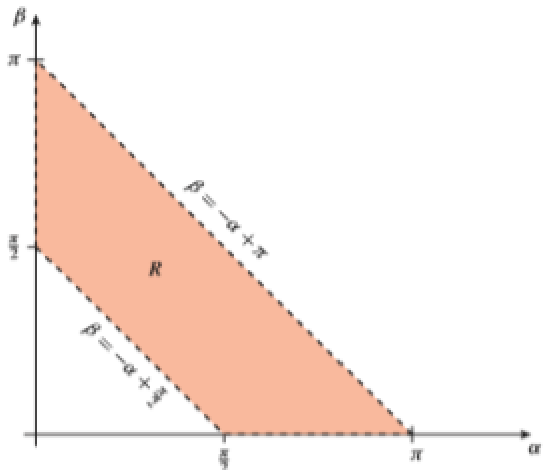

Figure 2 from [2]

The authors showed that $a + b > c + h$ fails exactly when $tan(\frac{\alpha}{2}) + tan(\frac{\beta}{2}) \leq 1$.And so $\beta \leq 2arctan(1 - tan(\frac{\alpha}{2}))$.The number of triangles that fails in *R* is shown below.

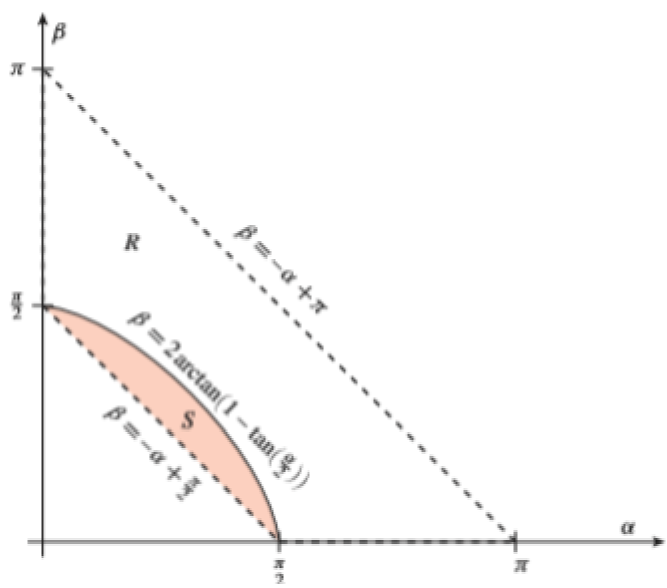

Figure 3 from [2]

The authors used *Maple* calculate the integral to determine that if S is the subset of R consisting of points where the stronger triangle inequality fails, then

$\frac{area(S)}{area(R)} \approx .08.$

Thus, one can conclude that the stronger triangle inequality holds approximately 92% of the time.

What happens if one multiplies h by $\frac{1}{2}$?

It is natural to also discuss what happens if the value for h changes? For example, what happens if h is divided by 2?

$$a + b > c + \frac{h}{2}$$

(5)

Following a similar approach above:

$$D := (a + b)^2 - (c + \frac{h}{2})^2 > 0$$

$$= a^2 + 2ab + b^2 - c^2 - ch - \frac{h^2}{4}$$

$$= (a^2 + b^2 - c^2) + 2ab - ch - \frac{h^2}{4}$$

$$= 2abcos\theta + 2ab - absin\theta - \frac{(absin\theta)^2}{4c^2}$$

$$= ab(2 + 2cos\theta - sin\theta) - \frac{absin^2\theta}{4c^2}$$

Since $2abcos\theta = a^2 + b^2 - c^2,\ ch = absin\theta,\ and\ h^2 = \frac{(absin\theta)^2}{c^2}$

$$= ab(2 + 2cos\theta - sin\theta - \frac{absin^2\theta}{4a^2+4b^2-8abcos\theta})$$

Since $c^2 = a^2 + b^2 - 2abcos\theta$. Now let $g(\theta) = 2 + 2cos\theta - sin\theta,\ R = \frac{a}{b}$, divide by b^2

$$D = ab[g(\theta) - \frac{Rsin^2\theta}{4R^2+4-8Rcos\theta}]$$

So (5) holds if and only if

$$\frac{D(4R^2+4-8Rcos\theta)}{ab} = 4g(\theta)R^2 - [8g(\theta)cos\theta + sin^2\theta]R + 4g(\theta)$$

$$\text{or } F(\theta, R) = 4g(\theta)[R^2 - 2Rcos\theta + 1] - Rsin^2\theta$$

As before, to find key points, set $g(\theta) = 0$

$$0 = 2 + 2cos\theta - sin\theta$$

$$(sin\theta)^2 = (2 + 2cos\theta)^2$$

$$1 - cos^2\theta = 4cos^2\theta + 4cos\theta + 4$$

$$0 = 5cos^2\theta + 4cos\theta + 3$$

And so $D = 64 - (4)(15) = 4$this yields:

$cos\theta = \frac{-8\pm2}{10}$so $cos\theta =- 1 \; or \; cos\theta =- \frac{3}{5}$

Therefore $\theta = \pi \; or \; \theta = \pi - arctan\frac{4}{3}$

When $g(\theta) \neq 0,$ then $F(\theta, R)$is a quadratic in R. Our discriminant:

$$D = [8g(\theta)cos\theta + sin^2\theta]^2 - 64g(\theta)^2$$

$$= [8g(\theta)cos\theta + sin^2\theta - 8g(\theta)][8g(\theta)cos\theta + sin^2\theta + 8g(\theta)]$$

(6)

Focusing on the left side::

$$0 = 8(2 + 2cos\theta - sin\theta)cos\theta + sin^2\theta - 8(2 + 2cos\theta - sin\theta)$$

$$0 = 16 + 16cos^2\theta - 8sin\theta cos\theta + sin^2\theta - 16 - 16cos\theta + 8sin\theta$$

The first and fifth terms cancel out.

The second and sixth terms become $- 15sin^2\theta$

$$0 =- 8sin\theta cos\theta - 15sin^2\theta + 8sin\theta$$

$$0 = 8sin\theta(1 - cos\theta) - 15(1 - cos\theta)(1 + cos\theta)$$

$$0 = (1 - cos\theta)(8sin\theta - 15cos\theta - 15)$$

And so $cos\theta = 1, cos\theta = \frac{-161}{289}$,or $cos\theta =- 1$

therefore $\theta = 0,\ \pi,\ or\ \pi - arctan\frac{240}{161}$

Focusing on the right side of (6):

$$0 = 16cos\theta + 16cos^2\theta - 8sin\theta cos\theta + sin^2\theta + 16 + 16cos\theta - 8sin\theta$$

The first two terms become $(1 + cos\theta)(16cos\theta)$.

The third and seventh terms become $(1 + cos\theta)(- 8sin\theta)$

The fourth term becomes $(1 - cos^2\theta) = (1 + cos\theta)(1 - cos\theta)$

The fifth and sixth terms become $16(1 + cos\theta)$

Together this yields:

$$0 = (1 + cos\theta)(16cos\theta + 1 - cos\theta - 8sin\theta + 16)$$

$$0 = (1 + cos\theta)(15cos\theta - 8sin\theta + 17)$$

And $cos\theta =- 1,\ or\ cos\theta =- \frac{15}{17}$

therefore $\theta = \pi\ or\ \pi - arctan\frac{8}{15}$.

By substituting the values back into $F(\theta, R)$ and solving for R, the following points of interest can be found: $P_1(0, 1)$, $P_2(\pi - arctan\frac{4}{3}, 0)$, $P_3(\pi - arctan\frac{8}{15}, - 1)$,$P_4(\pi, - 1)$,and $P_5(\pi - arctan\frac{240}{161}, 0.874)$with the line $\theta = \pi$.

The zeros of the discriminant occur at the four points $P_1(0, 1)$, $P_3(\pi - arctan\frac{8}{15}, - 1)$, $P_4(\pi, - 1)$,$P_5(\pi - arctan\frac{240}{161}, 0.874)$. Figure 6 shows the function in the theta-R plane. The upper and lower branches of the quadratic meet at P_2, P_3, P_4, and P_5. Notice the asymptote at $\theta = \pi - arctan\frac{4}{3}$,where all triangles with an angle greater than$\pi - arctan\frac{4}{3}$are not satisfied. All triangles where $\theta < arctan \frac{240}{161}$are satisfied, which are most triangles that have an angle less than $\pi - arctan\frac{4}{3}$.An isosceles triangle with the distinguished angle $\theta = arctan \frac{240}{161}$provides a counterexample to (5) with the smallest possible value of θ.But once again, what does “most” mean?

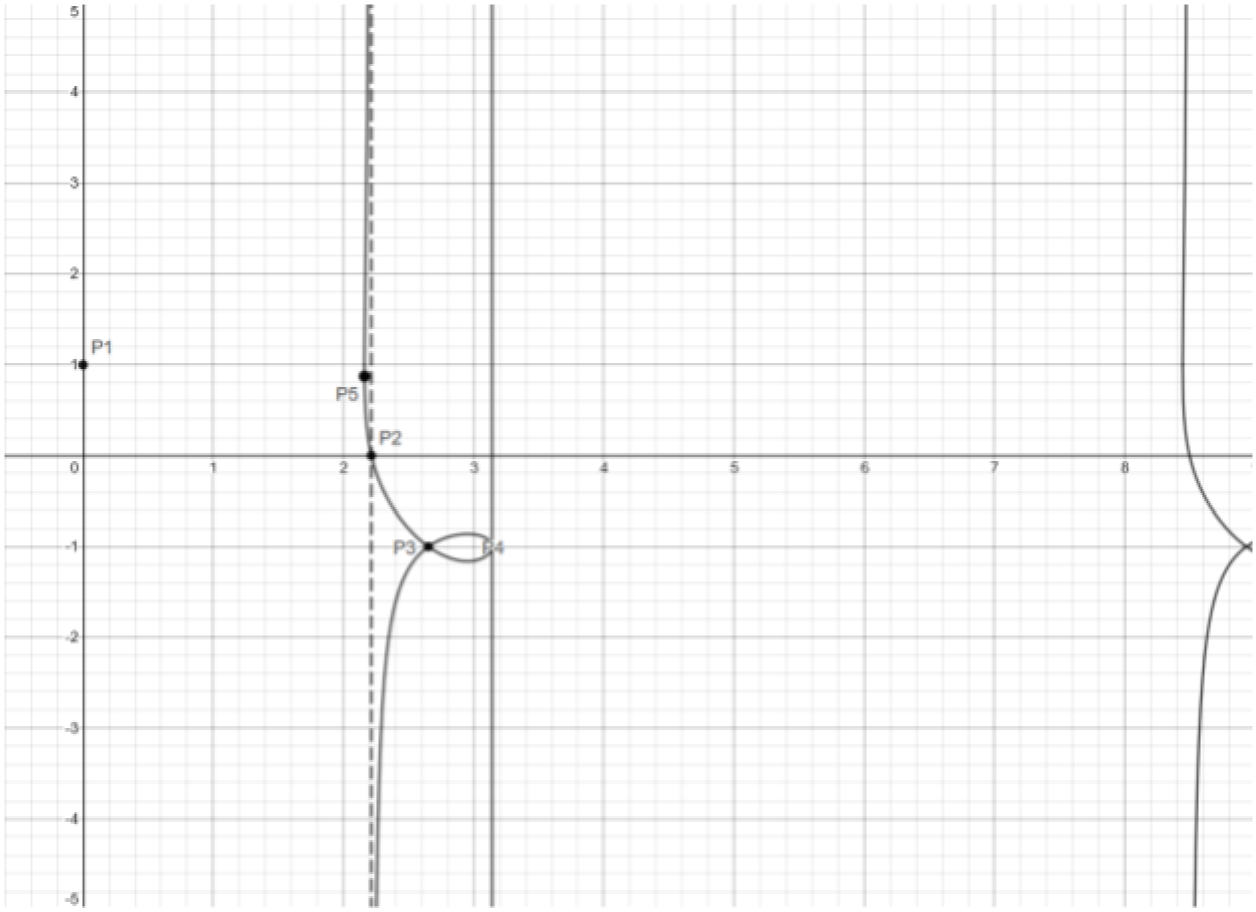

Figure 6

Percent of triangles satisfied

One can easily use a similar method that was used in [2] as shown earlier. Let $\theta = \pi - arctan\,\frac{4}{3}$,and given an ordered pair (α, β),which represents the two other angles in the triangle, then $\theta + \alpha + \beta = \pi$.The validity of the inequality does not change with respect to similarity, so all triangles with $\theta < \pi - arctan\,\frac{4}{3}$ are those with ordered pairs $(\alpha, \beta) \in R^2$ such that:

$0 < \alpha < \pi, 0 < \beta < \pi$, and so $\alpha + \beta > arctan\frac{4}{3}$ which yields

$$-\alpha + arctan\,\frac{4}{3} < \beta < -\alpha + \pi.$$

$$\text{So } \beta = -\alpha + arctan\,\frac{4}{3} \text{ and } \beta = -\alpha + \pi.$$

The following is a sketch of *R*.

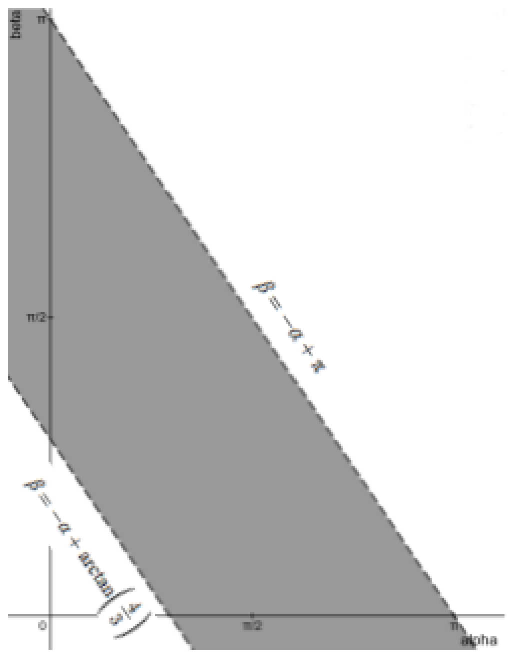

Figure 7

Let *S* be the subset of *R* consisting of points where (5) fails. Using a similar approach to Faiziev, Powers and Sahoo, (5) fails exactly when $tan(\frac{\alpha}{2}) + tan(\frac{\beta}{2}) \leq \frac{1}{2}$.This yields $\beta \leq 2arctan(\frac{1}{2} - tan(\frac{\alpha}{2}))$.A sketch of where it fails is highlighted in red below in figure 8.

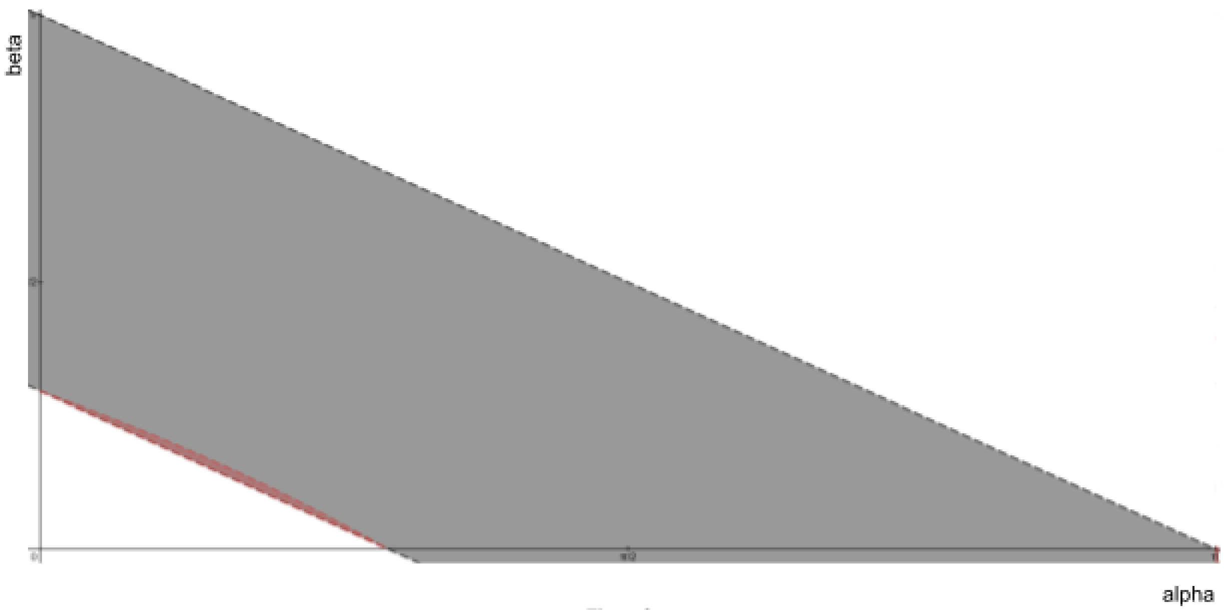

Figure 8

Using Desmos to calculate the integral:

$$\int_{0}^{arctan(\frac{4}{3})} (2arctan(\frac{1}{2} - tan(\frac{x}{2})) + x - arctan(\frac{4}{3}))\, dx \approx 0.3239$$

$$\text{Area of } R = \frac{1}{2}\pi^2 - \frac{1}{2}(arctan\frac{4}{3})^2$$

Therefore $\frac{area(S)}{area(R)} \approx 0.0071$

Thus, the stronger triangle inequality holds approximately 99.3% of the time when h is divided by two.

Conclusion

This study was able to explain, in further detail, the arguments made by Bailey and Banister when exploring the stronger triangle inequality $a + b > c + h$, showing how one can factor $F(\theta, R)$ without a CAS. The word “most” was also revisited for what it meant when trying to determine how many triangles with an angle less than $\frac{\pi}{2}$ satisfied such an inequality. It was then explored as to what would happen if the stronger triangle was manipulated and h was multiplied by $\frac{1}{2}$. Not only was a new distinguished angle discovered, but the study was also able to show how many triangles satisfy the new inequality with an angle that was less than $\pi - arctan(\frac{4}{3})$. It remains to be explored as to how one can find points of interest with a general k such that $0 < k < 1$in exploring the inequality $a + b > c + hk$.

References

1. H. R. Bailey and R. Bannister, A Stronger Triangle Inequality, *College Math. J.* **28** (1997) 182–186.

2. V. Faiziev, R. Powers and P. Sahoo. When Can One Expect a Stronger Triangle Inequality? *College Math. J.* **44** (2013) 24-31

3. M. S. Klamkin, A Sharp Triangle Inequality, *College Math. J.* **29** (1998) 33.